Comment on "Free surface Hele-Shaw flows around an obstacle: A random walk simulation"

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It is shown that the computer simulations of Hele-Shaw flows around a wedge reported by Bogoyavlenskiy and Cotts [Phys. Rev. E **69**, 016310 (2004)] do not reproduce with a high degree of accuracy the exact solutions known for this problem.

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Recently, Bogoyavlenskiy and Cotts [1] reported computer simulations of pressure driven Hele-Shaw flows where a more viscous fluid (liquid) advances against a less viscous fluid (gas) in the presence of a solid obstacle. Among the various different obstacles considered, they first analyzed the case of an infinite wedge of angle θ, as shown in the left panel of Fig. 1. The problem of Hele-Shaw flows around a wedge was previously studied by Cummings [2] and Richardson [3], where exact solutions were obtained. However, the numerical results reported in Ref. [1] for the wedge geometry differ considerably from the exact solutions obtained in Ref. [3] for a similar problem. As a possible explanation for these discrepancies, the authors of Ref. [1] argued that the problem they considered was somewhat different than that studied by Richardson [3]. The aim of the present Comment is to show that the exact solutions first obtained in Ref. [3], and rederived below with an alternative and more direct method, correspond precisely to the wedge geometry studied numerically in Ref. [1]. The degree of discrepancy between the numerical results presented in Ref. [1] and the exact solutions will also be quantified.

As is well known, the problem of Hele-Shaw flows can be formulated in terms of conformal mappings. Let $z=f(\zeta,t)$ be then the conformal mapping that maps the interior of the unit semicircle in the ζ complex plane into the physical fluid domain in the z plane, such that $\zeta=0$ corresponds to wedge vertex, at $z=0$, and $\zeta=-1$ corresponds to $z=\infty$, with the unit semicircle being mapped to the liquid-gas interface; see Fig. 1. For Hele-Shaw flows the complex potential $W(z,t)=\phi(x,y,t)+i\psi(x,y,t)$ is such that

$$\phi(x,y,t) = -\frac{b^2}{12\mu}p(x,y,t), \tag{1}$$

where p is the fluid pressure, b is the thickness of the Hele-Shaw cell, and μ is the fluid viscosity. In an abuse of notation, let us write $W(\zeta,t)\equiv W[f(\zeta),t]$. Then the complex potential $W(\zeta,t)$ must be an analytic function within the unit semicircle and satisfy the following boundary conditions:

$$\mathrm{Re}\, W = 0 \quad \text{on } |\zeta| = 1, \tag{2}$$

$$\mathrm{Im}\, W = 0 \quad \text{on } \zeta \in (-1,1), \tag{3}$$

$$\mathrm{Re}\left[\zeta W_\zeta - \frac{z_t \bar{z}_\zeta}{\zeta}\right] = 0 \quad \text{on } |\zeta| = 1. \tag{4}$$

In the last expression subscripts indicate partial derivatives and the bar denotes complex conjugate. Equation (2) follows from the fact that $p=0$ on the liquid-gas interface, whereas Eq. (3) ensures that the fluid normal velocity is zero at the solid surfaces, and Eq. (4) corresponds to the kinematic boundary condition [4]. Furthermore, it is assumed that far away from the wedge we have a uniform flow

$$W(z,t) \approx v_\infty z \quad \text{as } |z| \to \infty, \tag{5}$$

where v_∞ is a real constant.

From the preceding discussion it is clear that the flow domain in the complex W plane corresponds to the second quadrant $\phi \leqslant 0$, $\psi \geqslant 0$. It is then easy to verify that the conformal mapping from the interior of the unit semicircle in the ζ plane into the flow region in the W plane is effected by the following function:

$$W(\zeta,t) = C\left(\frac{\zeta-1}{\zeta+1}\right), \tag{6}$$

where C is a time-dependent parameter to be determined later. Now, the conformal mapping $z=f(\zeta,t)$ that maps the real diameter of the unit semicircle into the wedge solid surfaces and has the appropriate behavior at $\zeta \to -1$ (corresponding to $|z| \to \infty$) is given by

$$f(\zeta,t) = K e^{i\theta} \frac{\zeta^\alpha}{1+\zeta}, \tag{7}$$

where $\alpha=(\pi-\theta)/\pi$ and K is a real time-dependent parameter to be determined shortly. By analyzing the asymptotic

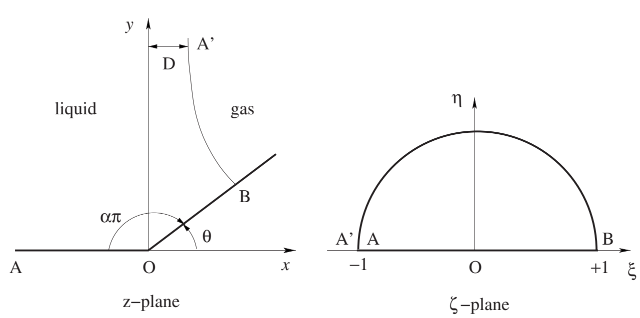

FIG. 1. The z plane and ζ plane used in the analysis.

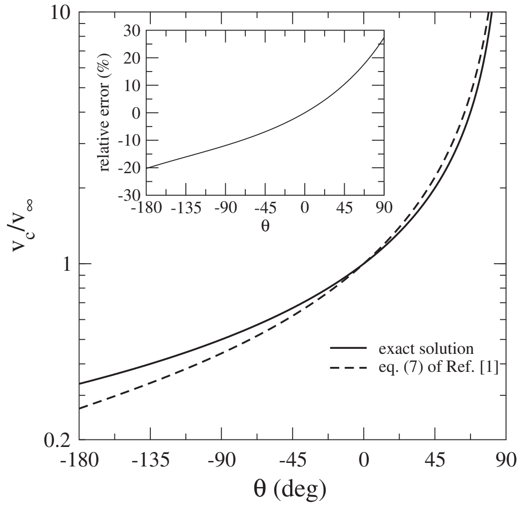

FIG. 2. Contact line velocity v_c (normalized to the flow velocity at infinity v_∞) as a function of the wedge angle θ. The solid line corresponds to the exact result given in Eq. (16), whereas the dashed line is a plot of Eq. (7) of Ref. [1]. The inset shows the relative error (in percent) between the latter and the former results.

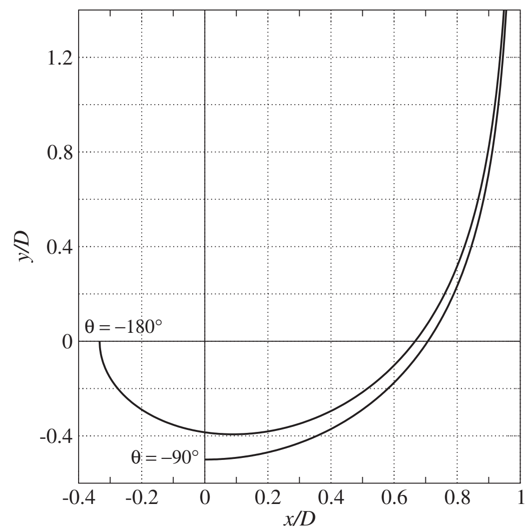

FIG. 3. The interface shape (normalized to the distance D) for $\theta=-90°$ and $\theta=-180°$.

behavior of $W(\zeta,t)$ and $f(\zeta,t)$ in the limit $\zeta\to-1$, one easily finds that condition (5) implies that

$$C=\frac{v_\infty K}{2}. \tag{8}$$

Similarly, one can show that in order to satisfy Eq. (4) one must have

$$\dot{K}=\frac{2v_\infty}{2\alpha-1}, \tag{9}$$

where the overdot denotes time derivative. Integration of the last equation then yields

$$K=\frac{2v_\infty t}{2\alpha-1}. \tag{10}$$

This determines the remaining parameter K and thus completes the solution. [I remark, parenthetically, that the mapping given by Eqs. (7) and (10) can be obtained as a particular case of a class of more general solutions found by Richardson [3]. The alternative method given above, in addition to being more direct, has the advantage that it is also suitable to treat Hele-Shaw flows around more complex obstacles, such as the step and needle geometries discussed in Ref. [1]. The difficulty in such cases, of course, is to find the appropriate mapping $f(\zeta,t)$. This problem is currently under investigation.]

The shape of the free surface can now be obtained by considering the image of the unit semicircle, $\zeta=e^{i\varphi}$, $0\leqslant\varphi<\pi$, under the mapping (7). After a simple calculation, one obtains the following parametric equations:

$$x=\frac{D}{2\alpha-1}\frac{\sin[(2\alpha-1)s]}{\sin s}, \tag{11}$$

$$y=\frac{D}{2\alpha-1}\frac{\cos[(2\alpha-1)s]}{\sin s}, \tag{12}$$

where $s=(\pi-\varphi)/2$, with $0<s\leqslant\pi/2$, and

$$D=\frac{1}{2}(2\alpha-1)K. \tag{13}$$

As Eq. (11) indicates, the parameter D represents the perpendicular distance between the asymptote of the free free boundary and the free boundary at the instant it touched the vertex; see Fig. 1. Comparing Eqs. (13) and (10) one sees that $D=v_\infty t$, as expected, since far away from the wedge the free surface becomes a flat interface that ought to move with the flow velocity at infinity. Equations (11)–(13) then show that the evolution of the interface for the wedge geometry is described by a similarity solution of the form

$$\frac{y}{v_\infty t}=f\left(\frac{x}{v_\infty t}\right), \tag{14}$$

where the function $f(x)$ can in principle be obtained from Eqs. (11) and (12).

In Ref. [1], the authors argue that the solutions for the wedge geometry studied in Ref. [3] and rederived above "deal with a boundary condition [at infinity] somewhat different from" the boundary condition they used. The derivation given above shows clearly that the boundary condition that is being fixed at infinity for the analytic solutions is precisely the same boundary condition used in the computer simulations, namely, a uniform flow far upstream, which in terms of the velocity potential reads

$$\phi(x,y,t)\approx v_\infty x \quad \text{as } x\to-\infty, \tag{15}$$

or as given in Eq. (5) in terms of the complex potential. Hence Eqs. (11) and (12) solve the very mathematical problem of Hele-Shaw flows around a wedge considered in Ref. [1].

The numerical results reported in Ref. [1] for the wedge geometry show, however, significant discrepancies from the exact solutions. For example, it follows from Eqs. (11) and (12) that the distance r_c from the wedge vertex to the point where the liquid-gas interface meets the solid surface is $r_c = D/(2\alpha - 1)$, so that the velocity v_c of the contact point depends on the wedge angle θ according to the expression

$$\frac{v_c}{v_\infty} = \frac{1}{2\alpha - 1} = \frac{1}{1 - 2\theta/\pi}. \tag{16}$$

Instead of this simple relationship, a rather complicated formula is proposed in Ref. [1] for the dependence of v_c on θ; see Eq. (7) of Ref. [1]. For comparison, I plot in Fig. 2 the contact line velocity v_c as a function of θ as predicted by both the exact result shown in Eq. (16) and the corresponding formula given in Eq. (7) of Ref. [1]. The inset of Fig. 2 shows the relative error between the later and the former expressions, which ranges from −20 to +27 % as θ varies from −180° to 90°.

The interfaces found numerically in Ref. [1] for the wedge geometry also show some discrepancies when compared to the analytical solutions. To illustrate this, I plot in Fig. 3 the free surface (rescaled by the distance D) as given by Eqs. (11) and (12) for $\theta = -90°$ and $\theta = -180°$. A direct comparison between this figure and Fig. 4 of Ref. [1] reveals noticeable differences between the numerical results and the exact solutions.

As a concluding remark, I should like to point out that the constructive method used to derive Eqs. (11) and (12) shows that this is the only possible form of similarity solution for the wedge geometry. (In terms of analytic functions this corresponds to the uniqueness of the solution to the mixed boundary value problem with given singularities.) Hence any time-dependent solution with a similarity-solution limit must tend to Eqs. (11) and (12). It thus remains unclear why the similarity solutions found numerically in Ref. [1] are in disagreement with the corresponding analytic solutions.

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